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# Discos relativistas constituidos por dos fluidos perfectos cargados eléctricamente

# Relativistic disks with two charged perfect fluids components

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A method to describe exact solutions of the Einstein-Maxwell field equations in terms of relativistic thin disks constituted by two perfect charged fluids is presented. Describing the surface of the disk as a single charged fluid we find explicit expressions for the rest energies, the pressures and the electric charge densities of the two fluids. An explicit example is given. The particular case of the thin disks composed by two charged perfect fluids with barotropic equation of state is also presented.

Se presenta un método para interpretar soluciones exactas de las ecuaciones de Einstein-Maxwell en términos de discos delgados relativistas constituidos por dos fluidos perfectos cargados eléctricamente. Mediante la descripción de la superficie del disco en términos de un único fluido cargado eléctricamente se encuentran expresiones explícitas para las energías en reposo, las presiones y las densidades de carga eléctica de los dos fluídos que lo componen. Como un ejemplo se presenta un caso particular de discos delgados compuestos por dos fluidos perfectos cargados descritos por una ecuación de estado barotrópica.

**KEYWORDS**

Relativistic disks, Einstein-Maxwell equations, Exact solutions, Relativistic fluids

**PALABRAS CLAVES**

Discos relativistas, Ecuaciones de Einstein-Maxwell, Soluciones exactas, Fluidos relativistas

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## I | INTRODUCTION

The problem of constructing exact models in general relativity to describe axially symmetric sources from the astrophysical point of view remains a topical problem. In particular, finding exact solutions of the Einstein-Maxwell field equations is a difficult task due to the non-linear character of the equations, in fact there are only a handful of physically acceptable solutions. In order to construct the models and obtain the solutions, one can follow two approaches. On the one hand, an equation of state is given and then the field equations are solved. On the other hand by means of the inverse method, where a solution

of the field equations is taken and then the energy-momentum (EM) tensor is obtained (González and Gutiérrez-Piñeres, 2012). For both cases, one has to pose correctly and solve for the exterior and the interior given boundary conditions for the Einstein-Maxwell equations. To obtain physically reasonable models the solutions must satisfy additional conditions compatible with the observable nature.
The energy-momentum tensor obtained using the inverse method has not a direct physical interpretation. Hence, to understand the physical meaning of the EM tensor, the eigenvalue problem should be solved. Alternatively, a suitable tetrad can be used to establish a relationship between the physical features of the tensor obtained from the inverse method and those of a well-known energy-momentum tensor. In most of the cases, the physical interpretation is difficult to accomplish. The inverse method has led to solutions of the Einstein-Maxwell equations in terms of thin disk models whose energy-momentum tensor can be interpreted as that of a single charged dust fluid, see for example (González et al., 2008; Gutiérrez-Piñeres et al., 2013) and references therein. The use of two fluids models to describe the surface energy-momentum tensor of relativistic disks and the understanding of the effects caused by the motions of each fluid component, can provide insight into the description of the sources for astrophysical objects. Two component anisotropic fluids have been studied in (Krisch and Glass, 2013; Letelier, 1980) and with multi-fluid components in (Letelier and Alencar, 1986). The problem of the relativistic thin disks constituted by only two non-interacting fluids has been explored in (Gutiérrez-Piñeres et al., 2015).
Inspired by the work of Letelier (Letelier, 1980), we present a simple method to interpret the material source of the relativistic thin disks in terms of two charged-perfect-fluids. As a direct application of the method, we study the solution to the Einstein-Maxwell equations for the thin disk-halos obtained in (Gutiérrez-Piñeres et al., 2015). This paper is organized as follows. First, in section II we introduce the two charged perfect fluid components method, where the surface energy-momentum tensor and the surface electric current density of the relativistic thin disk can be interpreted as a two perfect charged fluids. In the following section III we apply the method here presented to the solutions of the Einstein-Maxwell equations corresponding to thin disk-halos systems, as those in (Gutiérrez-Piñeres et al., 2015), to describe relativistic thin disks in terms of two charged perfect fluids. Finally, we present some concluding remarks in section IV.

## II | THE TWO CHARGED PERFECT FLUID COMPONENTS METHOD

In this section, we present a method for interpreting the solutions of the Einstein-Maxwell field equations corresponding to disk-like configurations of matter in terms of two charged perfect fluid components. To do so, we first assume that the surface energy-momentum tensor (SEMT) and the surface electric current density (SECD) of the disk can be written in the following form

$$S^{ab}(u,v) = t^{ab}(u) + t^{ab}(v),, \quad (1a)$$
$$J^{a}(u,v) = J^{a}(u) + J^{a}(v), \quad (1b)$$

where the tensors $t^{ab}(u)$ and $t^{ab}(v)$ and the 4-vectors $J^a(u)$ and $J^a(v)$ are defined as

$$t^{ab}(u) = (p+w)u^a u^b + ph^{ab}, \quad (2a)$$
$$t^{ab}(v) = (q+e)v^a v^b + qh^{ab}, \quad (2b)$$
$$J^a(u) = \eta u^a, \quad (2c)$$
$$J^a(v) = \lambda v^a. \quad (2d)$$

The SEMT of the disk (1a) is the sum of the surface energy-momentum tensor of two perfect fluids with pressures $p$ and $q$ and rest energies $w$ and $e$, respectively; $h^{ab}$ is the induced metric on the surface of the disk. Similarly, the SECD of the disk (1b) is the sum of two surface electric charge densities $\eta$ and $\lambda$ flowing with 4-velocities $u$ and $v$ on the surface of the disk. The 4-velocities of the fluids satisfy the condition

$$u^a u_a = v^a v_a \quad = \quad -1, \qquad u^a \neq v^a, \tag{3}$$

that is, the two time-like fluid velocities are not necessarily aligned. This can lead to interesting transport effects (Lopez-Monsalvo and Andersson, 2011) and to the generation of dynamical instabilities (Samuelsson et al., 2010). Now, we analyze the physical meaning of the SEMT and the SECD (1). To this end, we introduce the transformation

$$u^a \longrightarrow u^{*a} \quad = \quad u^a \cos\alpha + v^a \left(\frac{q+e}{p+w}\right)^{1/2} \sin\alpha, \tag{4a}$$

$$v^a \longrightarrow v^{*a} \quad = \quad -u^a \left(\frac{p+w}{q+e}\right)^{1/2} \sin\alpha + v^a \cos\alpha, \tag{4b}$$

in order to cast them in the general form of the SEMT and the SECD of a single fluid (Synge, 1964). It is easy to see that the SEMT is invariant under this transformation, then we can write this as

$$S^{ab} \quad = \quad (p+w)u^{a*}u^{*b} + (q+e)v^{*a}v^{*b} + (p+q)h^{ab}. \tag{5}$$

whereas under this transformation the surface electric current density is

$$J^a = q_1 u^{*a} + q_2 v^{*a}. \tag{6}$$

Here $q_1$ and $q_2$ are the surface electric charge densities of the fluids in the "new frame". The rotation angle $\alpha$ in (4) is fixed in terms of the velocity overlap by requiring $u^{*a}v^*_a = 0$, therefore, we may choose $u^{a*}$ as a time-like vector and $v^{*a}$ as a space-like vector. Equation (4) thus leads to

$$\tan 2\alpha = \frac{(p+w)^{1/2}(q+e)^{1/2}(2u^a v_a)}{(q+e)-(p+w)}. \tag{7}$$

We are now able to write down the (SEMT) and the (SECD) (1) in the general form of the energy-momentum tensor and electric current density for a *single fluid* flowing with 4-velocity $U^a$,

$$S^{ab} \quad = \quad \rho U^a U^b - \tau^{ab}, \tag{8a}$$

$$J^a \quad = \quad \psi U^a, \tag{8b}$$

with

$$\tau^{ab} U_a \quad = \quad 0, \tag{9a}$$

$$U^a U_a \quad = \quad -1. \tag{9b}$$

where $\rho$ is the rest energy of the single fluid, $\tau^{ab}$ is called the *stress tensor* and $\psi$ is the surface electric charge density of the fluid. If we define the following unit vectors

$$U^a \quad \equiv \quad \frac{u^{*a}}{(-u^{*a}u^*_a)^{1/2}}, \tag{10a}$$

$$\chi^a \quad \equiv \quad \frac{v^{*a}}{(v^{*a}v^*_a)^{1/2}}, \tag{10b}$$

which satisfy the conditions $U^aU_a = -\chi^a\chi_a = -1$ and $U^a\chi_a = 0$, it is possible to define the quantities

$$\rho \equiv S^{ab}U_aU_b = -(p+w)u^{*a}u^*_a - (p+q), \tag{11a}$$

$$\gamma \equiv S^{ab}\chi_a\chi_b = +(q+e)v^{*a}v^*_a + (p+q), \tag{11b}$$

$$\psi \equiv -U^aJ_a = q_1(-u^{*a}u^*_a)^{1/2}, \tag{11c}$$

$$\pi \equiv p+q, \tag{11d}$$

where we have used the equations (8). Using that $S^{ab}(u,v) = S^{ab}(u^*,v^*)$, the surface energy-momentum tensor can be written as

$$S^{ab} = (\rho+\pi)U^aU^b + (\gamma-\pi)\chi^a\chi^b + \pi h^{ab}, \tag{12}$$

which can be expressed as the surface energy momentum tensor of a single fluid (8a) with rest energy $\rho$ and with stress tensor

$$\tau^{ab} = -(\gamma-\pi)\chi^a\chi^b - \pi(U^aU^b + h^{ab}). \tag{13}$$

Note that

$$\begin{aligned} \chi^aU_a &= 0, \\ \tau^{ab}U_a &= 0, \\ \tau^{ab}\chi_b &= -\gamma\chi^a. \end{aligned}$$

Again, by using the fact that $S^{ab}(u,v) = S^{ab}(u^*,v^*)$ and the equations (11), we obtain the quantities $\rho$ and $\gamma$ in terms of $p, w, q$ and $e$,

$$\rho = +\frac{1}{2}(w+e-p-q) + \frac{1}{2}\left\{(p+w-q-e)^2 + 4(p+w)(q+e)(u^av_a)^2\right\}^{1/2}, \tag{14a}$$

$$\gamma = -\frac{1}{2}(w+e-p-q) + \frac{1}{2}\left\{(p+w-q-e)^2 + 4(p+w)(q+e)(u^av_a)^2\right\}^{1/2}, \tag{14b}$$

where $w$, $p$, $e$ and $q$ are the quantities defined in (2). To write the SECD in the general form for the electric current density of a *single fluid*, we use the transformation (4) and the surface electric current density given by (6). Then we obtain

$$J^a = u^a\left[q_1\cos\alpha - q_2\left(\frac{p+w}{q+e}\right)^{1/2}\sin\alpha\right] + v^a\left[q_1\left(\frac{q+e}{p+w}\right)^{1/2}\sin\alpha + q_2\cos\alpha\right]. \tag{15}$$

Consequently, from the equations (1b) and (15) we have the following relation between the surface electric charge densities

$$(q+e)(q_1^2-\eta^2) + (p+w)(q_2^2-\lambda^2) = 0. \tag{16}$$

On the other side, by using this relation, (14a) and the definition of surface charge density given by (11c) we have

$$\psi = \left(-\frac{q_2^2}{q+e} + \frac{\eta^2}{p+w} + \frac{\lambda^2}{q+e}\right)^{1/2}(\rho-p-q)^{1/2}. \tag{17}$$

Now, in order to guarantee the invariance of the electric charge density we demand that

$$\psi = (-J^a(U)J_a(U))^{1/2} = (-J^a(u^*,v^*)J_a(u^*,v^*))^{1/2} = (-J^a(u,v)J_a(u,v))^{1/2}. \tag{18}$$

From this it follows that $q_2 = 0$, and that the surface charge density $\psi$ defined in (8b) becomes

$$\psi = \left(\eta^2 + \lambda^2 - 2\eta\lambda u^a v_a\right)^{1/2}. \tag{19}$$

We thus have the following relations among the surface electric charge densities, the pressures, the rest energies and the velocities of the fluids:

$$\psi = \left(\frac{\eta^2}{p+w} + \frac{\lambda^2}{q+e}\right)^{1/2} (\rho - p - q)^{1/2}, \tag{20a}$$

$$2u^a v_a = \frac{\eta(2p+w+q-\rho)}{\lambda(p+w)} + \frac{\lambda(2q+e+p-\rho)}{\eta(q+e)}. \tag{20b}$$

In sum, we are equipped with a set of useful relations from which we can interpret the sources of the relativistic thin disk in terms of two charged perfect fluids. As we know, through the inverse method, we can always have explicit expressions for the surface energy momentum tensor and the surface electric current density of the relativistic disks. Then, we can always choose a suitable tetrad in terms of which the equations (7), (14) and (20) can be used to express the rest energy, the pressures, the velocity and the electric charge density of a single charged fluid characterizing the relativistic disk, in terms of two charged perfect fluids. In the next section we present a simple example to illustrate the application of the method developed in this section.

## III | RELATIVISTIC DISK-HALOS WITH TWO CHARGED PERFECT FLUIDS COMPONENTS

In order to illustrate the application of the method presented in the previous section to specific thin disks solutions, we will consider the solutions of the Einstein-Maxwell equations corresponding to thin disk-halos recently presented in (Gutiérrez-Piñeres et al., 2015). This kind of disks are constituted by a single charged dust fluid surrounded by a halo in presence of an electromagnetic field. The metric tensor is given by the conformastatic line element (Synge, 1964)

$$ds^2 = -e^{2\phi}dt^2 + e^{-2\beta\phi}[r^2 d\varphi^2 + dr^2 + dz^2]. \tag{21}$$

where $\phi$ depends on $r$ and $z$ only. Here, we only analyze the material source on the surface of the disk. First, we will cast the surface energy momentum tensor and the surface electric current density of the disks in terms of the locally static observer (LSO), that is

$$S_D^{ab} = \varepsilon e_{(0)}^{\ a} e_{(0)}^{\ b} + \wp e_{(1)}^{\ a} e_{(1)}^{\ b} + \wp e_{(2)}^{\ a} e_{(2)}^{\ b}, \tag{22a}$$

$$J_D^a = -\sigma e_{(0)}^{\ a}, \tag{22b}$$

where the non-zero components of the LSO are given by

$$e_{(0)}^{\ a} = e^{-\phi}\delta_0^a, \tag{23a}$$

$$e_{(1)}^{\ a} = r^{-1}e^{\beta\phi}\delta_1^a, \tag{23b}$$

$$e_{(2)}^{\ a} = e^{\beta\phi}\delta_2^a, \tag{23c}$$

$$e_{(3)}^{\ a} = e^{\beta\phi}\delta_3^a, \tag{23d}$$

being $\delta$ the usual Kronecker's delta symbol. With respect to the LSO, the components of the surface energy momentum tensor can be interpreted as the rest energy density and the pressures of the disk, given by the expressions

$$\begin{aligned}
\varepsilon(r) &= 4\beta m a F(r), && (24a)\\
\wp(r) &= \aleph\rho(r), && (24b)\\
F(r) &= \frac{(r^2+a^2)^{-\frac{2+\beta}{2+2\beta}}}{(1+\beta)\left(\sqrt{r^2+a^2}-m\right)^{\frac{2\beta+1}{1+\beta}}}. && (24c)
\end{aligned}$$

while the surface charge density of the disks is

$$\sigma(r) = \frac{2k_1 m a}{k\left(\sqrt{r^2+a^2}-m\right)^{\frac{2\beta}{1+\beta}}(r^2+a^2)^{\frac{3+\beta}{2(1+\beta)}}} \tag{25}$$

with $k_1$, $a$, $m$ and $k$ arbitrary constants and $\aleph = (1-\beta)/2\beta$. The parameter $\beta$ produces a non-zero pressure and it has the same value in the radial and angular directions. Now we rewrite (12) to get

$$S^{ab} = \rho U^a U^b + (\gamma - \pi)\chi^a\chi^b + \pi(h^{ab} + U^a U^b). \tag{26}$$

Comparing (22a) and (26), we find that the induced metric on the surface of the disk can be expressed as

$$h^{ab} \equiv g^{ab} - e_{(3)}^{\ a} e_{(3)}^{\ b} = -e_{(0)}^{\ a} e_{(0)}^{\ b} + e_{(1)}^{\ a} e_{(1)}^{\ b} + e_{(2)}^{\ a} e_{(2)}^{\ b}, \tag{27}$$

and that the vectors $U^a$ and $\chi^a$ given by (10) can be expressed in terms of the tetrad of the LSO as

$$\begin{aligned}
U^a &\equiv e_{(0)}^{\ a}, && (28a)\\
\chi^a &\equiv e_{(1)}^{\ a}. && (28b)
\end{aligned}$$

By inserting (27) and (28) into (26) we get the expression for the surface energy momentum tensor

$$S^{ab} = \rho e_{(0)}^{\ a} e_{(0)}^{\ b} + \gamma e_{(1)}^{\ a} e_{(1)}^{\ b} + \pi e_{(2)}^{\ a} e_{(2)}^{\ b}, \tag{29}$$

while, with $U^a$ given by (28a) the surface electric charge density (8b) may be rewritten as

$$J^a = \psi e_{(0)}^{\ a}. \tag{30}$$

Now, if we compare the expressions (29) and (30) (corresponding to the surface energy momentum tensor and the surface electric current density for a general relativistic disk) with the expression (22) (corresponding to the surface energy momentum tensor and the surface electric current density for the relativistic disk-halos presented in Gutiérrez-Piñeres et al. (2015)), we find that

$$\begin{aligned}
\varepsilon &= \rho = w + e, && (31a)\\
\wp_1 &= \wp_2 = \aleph\rho = p + q, && (31b)\\
-\sigma &= \psi = (1-\aleph)^{1/2}\left(\frac{\eta^2}{p+w} + \frac{\lambda^2}{q+e}\right)^{1/2}(w+e)^{1/2}, && (31c)\\
\gamma &= p + q, && (31d)\\
2u^a v_a &= \frac{\eta(2p+q)}{\lambda(p+w)} + \frac{\lambda(2q+p)}{\eta(q+e)}, && (31e)\\
\tan 2\alpha &= \frac{(p+w)^{1/2}(q+e)^{1/2} 2u^a v_a}{(q+e)-(p+w)}. && (31f)
\end{aligned}$$

Therefore, we can conclude from the last expression that, the rest energy $\varepsilon$ as well as the pressures of the disk $\wp = \wp_1 = \wp_2$, can be understood as the sum of the rest energies and pressures of the two charged perfect fluids, respectively. The relation (31e) says that the fluids are not necessarily aligned. The 4-velocity vectors of the fluids, $u^a$ and $v^a$, are both time-like and pointing towards the future, then, they satisfy the condition (Synge, 1964)

$$u^a v_a = k \leq -1, \tag{32}$$

with $k$ an arbitrary constant. These velocity vectors can be expressed in term of the LSO tetrad as

$$u^a = \cos(\alpha) A e_{(0)}^{\ a} - \frac{(q+e)^{1/2}}{(p+w)^{1/2}} \sin\alpha B e_{(1)}^{\ a}, \tag{33a}$$

$$v^a = \frac{(p+w)^{1/2}}{(q+e)^{1/2}} \sin\alpha A e_{(0)}^{\ a} + \cos\alpha B e_{(1)}^{\ a}, \tag{33b}$$

with the quantities $A$ and $B$ given by

$$A^2 = -u^{*a}u^*_a = \cos^2\alpha + \left(\frac{q+e}{p+w}\right)\sin^2\alpha - 2\left(\frac{q+e}{p+w}\right)^{1/2} \sin\alpha\cos\alpha u^a v_a, \tag{34a}$$

$$B^2 = v^{*a}v^*_a = -\cos^2\alpha - \left(\frac{p+w}{q+e}\right)\sin^2\alpha - 2\left(\frac{p+w}{q+e}\right)^{1/2} \sin\alpha\cos\alpha u^a v_a. \tag{34b}$$

The constraint

$$2\left(\frac{p+w}{q+e}\right)^{1/2} \cos\alpha\sin\alpha|k| \geq \cos^2\alpha + \left(\frac{p+w}{q+e}\right)\sin^2\alpha, \tag{35}$$

must be satisfied to guarantee that the vectors $u^a$ and $v^a$ are well-defined.

Now, for convenience, we could particularize our results for $p$, $q$, $w$ and $e$ by assuming that the pressures and the rest energies of the fluids are related by $p = k_1 q$, and $w = k_1 e$, $k_1$ being an arbitrary constant. In such case we obtain

$$\tan 2\alpha = \frac{2k_1^{1/2}|k|}{k_1 - 1}, \tag{36a}$$

$$p = \aleph w, \tag{36b}$$

$$q = \aleph e, \tag{36c}$$

and the electric charge densities of the fluid satisfy the relation

$$-2|k|k_1(\aleph + 1)\lambda\eta = \aleph[\eta^2(2k_1+1) + \lambda^2(2+k_1)k_1], \tag{37}$$

with the total surface electric charge given by the equation (31c).

We have presented a simple example to illustrate the method outlined in section II. In this case, we have described explicitly the surface energy momentum tensor and the surface electric current of the relativistic thin disk-halos discussed in the reference (Gutiérrez-Piñeres et al., 2015) as the sum of the surface energy momentum tensor of two charged perfect fluids with pressures $p$ and $q$ and rest energies $w$ and $e$, respectively. In a similar way, we have also expressed the surface electric current density of the disks as the sum of two surface electric current densities with charge densities $\eta$ and $\lambda$ flowing with velocities $u$ and $v$ on the surface of the disk. In the final part of this section we have particularized the calculations for a specific relation for the two pressures and between the rest energies of the constituting fluids. With this choice, we have obtained that the disk can be described as being constituted by two charged perfect fluids with a barotropic equation of state.

## IV | CONCLUDING REMARKS

In this work, we derived a method to interpret the solutions of Einstein-Maxwell field equations for relativistic thin disks in terms of two charged perfect fluids. The method was developed under the assumption that the surface energy momentum tensor as well as the electric current density of the disk can be expressed as the sum of the energy momentum tensors and the electric current densities of two charged perfect fluids. Moreover, we have obtained explicit relations that can be used to express the rest energy, the pressures and the electric charge density on the surface of the disk as a single charged fluid composed of two charged perfect fluids.
As an illustration of the application of the method, we have described the solutions of the Einstein-Maxwell field equations corresponding to the disk-halos discussed in the reference (Gutiérrez-Piñeres et al., 2015) in terms of two charged perfect fluids flowing on the surface of the disk with pressures $p$ and $q$ and rest energies $w$ and $e$, respectively. For a particular relationship between the pressures of the fluids, and between the rest energies of the fluids, we have obtained that the disks can be described as constituted by two charged perfect fluids with barotropic equation of state. Here we have restricted the method to the static case. We consider that the method studied in this work can be also used to describe stationary relativistic thin disks in terms of two charged perfect fluids. Work along this direction will soon be reported.

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